

On quantum L-operator for two-dimensional lattice Toda model

ANDREI BYTSKO,^{1,2} IRINA DAVYDENKOVA²

¹ Steklov Mathematics Institute
Fontanka 27, 191023, St. Petersburg, Russia

² Section of Mathematics, University of Geneva
2–4 rue de Lièvre, C.P. 64, 1211 Genève 4, Switzerland

Abstract

The two-dimensional quantum lattice Toda model for the affine and simple Lie algebras of the type A is considered. For its known L-operator a correction of the second order in the lattice parameter ε is found. It is proved that the equation determining a correction of the third order in ε has no solutions.

1 Introduction

1.1 Continuous classical model

The (1+1)-dimensional Toda chain associated with the affine Lie algebra $A_{N-1}^{(1)}$ is a model that describes relativistic dynamics of N scalar fields, ϕ_a , $a = 1, \dots, N$, assigned to the nodes of the corresponding Dynkin diagram. Their equations of motion are

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \phi_a = \frac{2m^2}{\beta} \left(e^{2\beta(\phi_{a+1}-\phi_a)} - e^{2\beta(\phi_a-\phi_{a-1})}\right). \quad (1)$$

Here and below the index which enumerates the nodes of the affine Dynkin diagram takes values in \mathbb{Z}/N . In particular, we have $\phi_{N+a} \equiv \phi_a$. Equations of motion (1) are generated by the following Hamiltonian and Poisson structure:

$$H = \sum_{a=1}^N \int dx \left(\frac{1}{2} \pi_a^2 + \frac{1}{2} (\partial_x \phi_a)^2 + \frac{m^2}{\beta^2} e^{2\beta(\phi_{a+1}-\phi_a)} \right), \quad (2)$$

$$\{\pi_a(x), \phi_b(y)\} = \delta_{ab} \delta(x-y). \quad (3)$$

The model under consideration is integrable. It admits the zero curvature representation with the following U - V pair [1, 2]

$$U(\lambda) = \sum_{a=1}^N \beta \pi_a e_{aa} + m \sum_{a=1}^N e^{\beta(\phi_{a+1}-\phi_a)} (\lambda^{\delta_{a,N}} e_{a,a+1} + \lambda^{-\delta_{a,N}} e_{a+1,a}), \quad (4)$$

$$V(\lambda) = \sum_{a=1}^N \beta \partial_x \phi_a e_{aa} + m \sum_{a=1}^N e^{\beta(\phi_{a+1}-\phi_a)} (\lambda^{\delta_{a,N}} e_{a,a+1} - \lambda^{-\delta_{a,N}} e_{a+1,a}), \quad (5)$$

where e_{ab} stands for the basis matrix such that $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

The matrix U satisfies the following relation (the so-called fundamental Poisson brackets, see [3]):

$$\{U_1(\lambda), U_2(\mu)\} = [r(\frac{\lambda}{\mu}), U_1(\lambda) + U_2(\mu)], \quad (6)$$

where $r(\lambda)$ is the classical trigonometric r -matrix for the algebra A_{N-1} , see [4, 5, 3]. Here and below the lower indices denote the tensor component, e.g. $U_1 = U \otimes \mathbb{I}$.

1.2 Quantum lattice model

The direct quantization of a continuous interacting field theory is known to have problems with ultraviolet divergences. A possible roundabout is to consider a discrete regularization of the model by putting it on the one dimensional lattice of a step Δ . For the lattice model, the quantum canonical variables that sit at different sites commute, and those that sit at the same site satisfy the following relations

$$[\pi_a, \phi_b] = -i \hbar \delta_{ab}. \quad (7)$$

The classical continuous limit of these relations recovers the Poisson structure (3) if one assumes that

$$\pi_a^{(n)} = \Delta \pi_a(x) \quad \phi_a^{(n)} = \phi_a(x), \quad x = n\Delta, \quad (8)$$

where n is the lattice site's number (it will be omitted in the subsequent formulae).

Given an integrable classical continuous model, its quantum lattice analogue is integrable as well if there exist a quantum L -operator (see, e.g. [6]) such that:

i) its classical continuous limit recovers the corresponding matrix U :

$$L(\lambda) \Big|_{\hbar=0} = \mathbb{I} + \Delta U(\lambda) + o(\Delta); \quad (9)$$

ii) it satisfies the following quadratic commutation relation which is a lattice analogue of the fundamental Poisson brackets (6):

$$R(\frac{\lambda}{\mu}) L_1(\lambda) L_2(\mu) = L_2(\mu) L_1(\lambda) R(\frac{\lambda}{\mu}). \quad (10)$$

The quantum R -matrix must satisfy the Yang-Baxter relation,

$$R_{12}(\frac{\lambda}{\mu}) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\frac{\lambda}{\mu}), \quad (11)$$

and its classical limit must recover the classical r -matrix.

For the $A_{N-1}^{(1)}$ Toda model, the quantum R-matrix is given by [4, 5]

$$R(\lambda) = \sum_{a,b=1}^N (\lambda q^{\delta_{ab}} - q^{-\delta_{ab}}) e_{aa} \otimes e_{bb} + (q - q^{-1}) \sum_{a \neq b}^N \lambda^{\theta_{ab}} e_{ab} \otimes e_{ba}, \quad (12)$$

where $q = e^{i\beta^2 \hbar}$, and $\theta_{ab} = 0$ for $a < b$, $\theta_{ab} = 1$ for $a > b$.

2 Lattice quantum L-operator

2.1 First order

We will use the following notations:

$$\begin{aligned} \Pi &= \text{diag}(\pi_1, \dots, \pi_N), & \Phi &= \text{diag}(\phi_1, \dots, \phi_N), \\ \hat{e}_a &= \lambda^{\delta_{a,N}} e_{a,a+1}, & \hat{f}_a &= \lambda^{-\delta_{a,N}} e_{a+1,a}, \\ \hat{E} &= \sum_{a=1}^N \hat{e}_a, & \hat{F} &= \sum_{a=1}^N \hat{f}_a. \end{aligned}$$

In the seminal work [5] M. Jimbo has found an approximate quantum L-operator for the $A_{N-1}^{(1)}$ Toda model. Namely, he showed that the following L-operator

$$\begin{aligned} L^J(\lambda) &= e^{\frac{\beta}{2}\Pi} \left(\mathbb{I} + \varepsilon (e^{-\beta \text{ad}_\Phi} \hat{E} + e^{\beta \text{ad}_\Phi} \hat{F}) \right) e^{\frac{\beta}{2}\Pi} \\ &= \sum_{a=1}^N e_{aa} e^{\beta \pi_a} + \varepsilon e^{\frac{\beta}{2}\Pi} \left(\sum_{a=1}^N e^{\beta(\phi_{a+1} - \phi_a)} (\hat{e}_a + \hat{f}_a) \right) e^{\frac{\beta}{2}\Pi} \end{aligned} \quad (13)$$

satisfies the RLL-relations (10) in the zeroth and first orders in ε .

It is easy to see that (13) satisfies the condition (9) if we set

$$\varepsilon = m\Delta \quad (14)$$

and take into account the “renormalization” of momenta (8) in the continuous limit.

Note that, although the L-operator (13) is approximate, the corresponding R-matrix (12) contains no small parameter ε and is an exact solution to (11). In order to treat the quantum Toda model by means of the quantum inverse scattering method (see [6]) one needs an exact quantum L-operator which solves the relation (10) in all orders in ε . In the present paper we will consider second and third order corrections to the L-operator (13).

2.2 Second order

Consider an L-operator $L(\lambda, \varepsilon)$ that admits a series expansion in the parameter ε ,

$$L(\lambda, \varepsilon) = \sum_{n \geq 0} \varepsilon^n L^{(n)}(\lambda). \quad (15)$$

Expanding relation (10) in ε , we obtain an infinite set of relations for $L^{(n)}(\lambda)$. The explicit form of those corresponding to the order ε^n , $n = 0, 1, 2, 3$ is

$$R(\frac{\lambda}{\mu}) L_1^{(0)}(\lambda) L_2^{(0)}(\mu) = L_2^{(0)}(\mu) L_1^{(0)}(\lambda) R(\frac{\lambda}{\mu}), \quad (16)$$

$$R(\frac{\lambda}{\mu}) (L_1^{(1)}(\lambda) L_2^{(0)}(\mu) + L_1^{(0)}(\lambda) L_2^{(1)}(\mu)) = \\ (L_2^{(1)}(\mu) L_1^{(0)}(\lambda) + L_2^{(0)}(\mu) L_1^{(1)}(\lambda)) R(\frac{\lambda}{\mu}), \quad (17)$$

$$R(\frac{\lambda}{\mu}) (L_1^{(2)}(\lambda) L_2^{(0)}(\mu) + L_1^{(0)}(\lambda) L_2^{(2)}(\mu) + L_1^{(1)}(\lambda) L_2^{(1)}(\mu)) = \\ (L_2^{(2)}(\mu) L_1^{(0)}(\lambda) + L_2^{(0)}(\mu) L_1^{(2)}(\lambda) + L_2^{(1)}(\mu) L_1^{(1)}(\lambda)) R(\frac{\lambda}{\mu}). \quad (18)$$

$$R(\frac{\lambda}{\mu}) (L_1^{(3)}(\lambda) L_2^{(0)}(\mu) + L_1^{(0)}(\lambda) L_2^{(3)}(\mu) + L_1^{(2)}(\lambda) L_2^{(1)}(\mu) + L_1^{(1)}(\lambda) L_2^{(2)}(\mu)) = \\ (L_2^{(3)}(\mu) L_1^{(0)}(\lambda) + L_2^{(0)}(\mu) L_1^{(3)}(\lambda) + L_2^{(2)}(\mu) L_1^{(1)}(\lambda) + L_2^{(1)}(\mu) L_1^{(2)}(\lambda)) R(\frac{\lambda}{\mu}). \quad (19)$$

We will take

$$L^{(0)}(\lambda) = e^{\beta\Pi}, \quad L^{(1)}(\lambda) = e^{\frac{\beta}{2}\Pi} (\rho_+ e^{-\beta\text{ad}_\Phi} \hat{E} + \rho_- e^{\beta\text{ad}_\Phi} \hat{F}) e^{\frac{\beta}{2}\Pi}. \quad (20)$$

Notice that we slightly generalized the first order L-operator (13) by introducing arbitrary coefficients ρ_+ , ρ_- . In order to comply with the classical limit condition (9) we have to assume that $\rho_+, \rho_- \rightarrow 1$ as $\hbar \rightarrow 0$.

The problem which we want to solve is the following. First, given $L^{(0)}(\lambda)$ and $L^{(1)}(\lambda)$ as in (20), find the most general solution $L^{(2)}(\lambda)$ to the equation (18). Then investigate whether, for some suitable $L^{(2)}(\lambda)$, equation (19) has a solution $L^{(3)}(\lambda)$.

The main result of the present article is the following statement:

Proposition 1. *Let $R(\lambda)$ be given by (12), and $L^{(0)}(\lambda)$, $L^{(1)}(\lambda)$ by (20). Then*

i) *The general solution to equation (18) is given by*

$$\tilde{L}^{(2)}(\lambda) = L^{(2)}(\lambda) + \tilde{L}^{(1)}(\lambda). \quad (21)$$

Here $\tilde{L}^{(1)}(\lambda)$ is an arbitrary solution to equation (17), and $L^{(2)}(\lambda)$ is given by

$$L^{(2)}(\lambda) = e^{\frac{\beta}{2}\Pi} \left(\gamma_1 e^{-\beta\text{ad}_\Phi} \hat{E}^2 + \gamma_2 e^{\beta\text{ad}_\Phi} \hat{F}^2 \right. \\ \left. + \gamma_3 (e^{-\beta\text{ad}_\Phi} \hat{E})(e^{\beta\text{ad}_\Phi} \hat{F}) + \gamma_4 (e^{\beta\text{ad}_\Phi} \hat{F})(e^{-\beta\text{ad}_\Phi} \hat{E}) \right) e^{\frac{\beta}{2}\Pi}, \quad (22)$$

where the coefficients γ_i must satisfy the following conditions

$$\text{for } N = 2: \quad \gamma_3 + \gamma_4 = \rho_+ \rho_-, \quad (23)$$

$$\text{for } N \geq 3: \quad \gamma_1 = \frac{q \rho_+^2}{1+q}, \quad \gamma_2 = \frac{\rho_-^2}{1+q}, \quad \gamma_3 + \gamma_4 = \rho_+ \rho_-. \quad (24)$$

ii) *For any choice of $\tilde{L}^{(1)}(\lambda)$ in (21), equation (19) has no solution for $L^{(3)}(\lambda)$.*

Proof is given in Appendix A.

Formula (21) reflects the fact that the general solution to an inhomogeneous equation is the sum of its particular solution and the general solution of the corresponding homogeneous equation. Let us remark that $\tilde{L}^{(1)}(\lambda)$ does not have to satisfy the condition (9).

The explicit expression for (22) involving the basis matrices is

$$\begin{aligned} L^{(2)}(\lambda) = & \sum_{a=1}^N e^{\frac{\beta}{2}\pi_a} (\gamma_3 e^{2\beta(\phi_{a+1}-\phi_a)} + \gamma_4 e^{2\beta(\phi_a-\phi_{a-1})}) e^{\frac{\beta}{2}\pi_a} e_{aa} \\ & + \gamma_1 \sum_{a=1}^N e^{\frac{\beta}{2}\pi_{a-1}} e^{\beta(\phi_{a+1}-\phi_{a-1})} e^{\frac{\beta}{2}\pi_{a+1}} e_{a-1,a+1} \lambda^{\delta_{a,1}+\delta_{a,N}} \\ & + \gamma_2 \sum_{a=1}^N e^{\frac{\beta}{2}\pi_{a+1}} e^{\beta(\phi_{a+1}-\phi_{a-1})} e^{\frac{\beta}{2}\pi_{a-1}} e_{a+1,a-1} \lambda^{\delta_{a,1}+\delta_{a,N}}. \end{aligned} \quad (25)$$

For $N = 2$, eq. (25) contains only diagonal terms. In this case, choosing $\rho_+ = \rho_- = 1$, $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 = 1$ (or $\gamma_3 = 0$), we obtain an exact L-operator,

$$L(\lambda) = \begin{pmatrix} e^{\frac{\beta}{2}\pi_1} (1 + \varepsilon^2 e^{2\tilde{\beta}(\phi_2-\phi_1)}) e^{\frac{\beta}{2}\pi_1} & \varepsilon e^{\frac{\beta}{2}\pi_1} (e^{\beta(\phi_2-\phi_1)} + \lambda^{-1} e^{\beta(\phi_1-\phi_2)}) e^{\frac{\beta}{2}\pi_2} \\ \varepsilon e^{\frac{\beta}{2}\pi_2} (e^{\beta(\phi_2-\phi_1)} + \lambda e^{\beta(\phi_1-\phi_2)}) e^{\frac{\beta}{2}\pi_1} & e^{\frac{\beta}{2}\pi_2} (1 + \varepsilon^2 e^{2\tilde{\beta}(\phi_1-\phi_2)}) e^{\frac{\beta}{2}\pi_2} \end{pmatrix}, \quad (26)$$

where $\tilde{\beta} = (\gamma_3 - \gamma_4)\beta$. L-operator (26) satisfies relation (10) in all orders in ε . Upon the reduction $\phi_2 = -\phi_1$, $\pi_2 = -\pi_1$, eq. (26) yields the well-known exact L-operator for the sinh-Gordon model [7, 6].

3 Reduction to non-affine case

The (1+1)-dimensional Toda chain associated with the simple Lie algebra A_{N-1} describes relativistic dynamics of N scalar fields whose equations of motion are given by the same equation (1) where no periodicity in the index a is assumed. In this case one can formally set $\beta\phi_0 = -\beta\phi_{N+1} = +\infty$ in (1) and (2). The same procedure applied to (4)–(5) yields the U–V pair without a spectral parameter.

In order to keep the spectral parameter in the U–V pair, the following procedure was suggested in [8] (in the case corresponding to A_1 ; a generalization was considered in [9]). Take $\xi > 0$ and shift (the zero modes of) the fields, the mass and the spectral parameter in (4)–(5) as follows,

$$\phi_a \rightarrow \phi_a + a\xi/\beta, \quad m \rightarrow e^{-\xi} m, \quad \lambda \rightarrow e^{\xi N} \lambda. \quad (27)$$

Then the limit $\xi \rightarrow +\infty$ yields the following U–V pair

$$U(\lambda) = \beta\Pi + m(e^{-\beta\text{ad}_\Phi} \hat{E} + e^{\beta\text{ad}_\Phi} F), \quad V(\lambda) = \beta\partial_x \Phi + m(e^{-\beta\text{ad}_\Phi} \hat{E} - e^{\beta\text{ad}_\Phi} F), \quad (28)$$

where

$$\hat{E} = \sum_{a=1}^N \hat{e}_a = \sum_{a=1}^N \lambda^{\delta_{a,N}} e_{a,a+1}, \quad F = \sum_{a=1}^{N-1} \hat{f}_a = \sum_{a=1}^{N-1} e_{a+1,a}. \quad (29)$$

The U-matrix in (28) satisfies the same fundamental Poisson bracket (6) with the same classical r-matrix as in the affine case.

In the non-affine case we have the following counterpart of Proposition 1.

Proposition 2. *Let $R(\lambda)$ be given by (12). Then*

i) Equations (16) and (17) admit the following solutions

$$L^{(0)}(\lambda) = e^{\beta\Pi}, \quad L^{(1)}(\lambda) = e^{\frac{\beta}{2}\Pi} (\rho_+ e^{-\beta\text{ad}_\Phi} \hat{E} + \rho_- e^{\beta\text{ad}_\Phi} F) e^{\frac{\beta}{2}\Pi}. \quad (30)$$

ii) Given $L^{(0)}(\lambda)$, $L^{(1)}(\lambda)$ as in (30), the general solution to equation (18) is given by

$$\tilde{L}^{(2)}(\lambda) = L^{(2)}(\lambda) + \tilde{L}^{(1)}(\lambda). \quad (31)$$

Here $\tilde{L}^{(1)}(\lambda)$ is an arbitrary solution to equation (17), and $L^{(2)}(\lambda)$ is given by

$$L^{(2)}(\lambda) = e^{\frac{\beta}{2}\Pi} \left(\gamma_1 e^{-\beta\text{ad}_\Phi} \hat{E}^2 + \gamma_2 e^{\beta\text{ad}_\Phi} F^2 + \gamma_3 (e^{-\beta\text{ad}_\Phi} \hat{E})(e^{\beta\text{ad}_\Phi} F) + \gamma_4 (e^{\beta\text{ad}_\Phi} F)(e^{-\beta\text{ad}_\Phi} \hat{E}) \right) e^{\frac{\beta}{2}\Pi}, \quad (32)$$

where the coefficients γ_i must satisfy conditions (23) and (24).

iii) For any choice of $\tilde{L}^{(1)}(\lambda)$ in (31), equation (19) has no solution for $L^{(3)}(\lambda)$.

Proof is given in Appendix.

For $N = 2$, eq. (32) contains only diagonal terms. Furthermore, $F^2 = 0$. In this case, choosing $\rho_+ = \rho_- = 1$, $\gamma_1 = 0$, and $\gamma_3 = 1$ (or $\gamma_3 = 0$), we obtain an exact L-operator,

$$L(\lambda) = \begin{pmatrix} e^{\frac{\beta}{2}\pi_1} (1 + \varepsilon^2 e^{2\tilde{\beta}(\phi_2 - \phi_1)}) e^{\frac{\beta}{2}\pi_1} & \varepsilon e^{\frac{\beta}{2}\pi_1} e^{\beta(\phi_2 - \phi_1)} e^{\frac{\beta}{2}\pi_2} \\ \varepsilon e^{\frac{\beta}{2}\pi_2} (e^{\beta(\phi_2 - \phi_1)} + \lambda e^{\beta(\phi_1 - \phi_2)}) e^{\frac{\beta}{2}\pi_1} & e^{\frac{\beta}{2}\pi_2} (1 + \varepsilon^2 e^{2\tilde{\beta}(\phi_1 - \phi_2)}) e^{\frac{\beta}{2}\pi_2} \end{pmatrix}, \quad (33)$$

where $\tilde{\beta} = (\gamma_3 - \gamma_4)\beta$. L-operator (33) satisfies relation (10) in all orders in ε . Upon the reduction $\phi_2 = -\phi_1$, $\pi_2 = -\pi_1$, eq. (33) yields the exact L-operator for the Liouville model [8].

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A Appendix

A.1 Proof of the Proposition 1. Second order

We will use the following notations:

$$H_a = e_{aa} - e_{a+1, a+1}, \quad K_a = q^{\frac{1}{2}H_a}, \quad \alpha_a(X) = \text{tr}(H_a X),$$

where $a = 1, \dots, N$ and $e_{N+1, N+1} \equiv e_{11}$. Then we have

$$K_a \hat{e}_b = q^{\frac{1}{2}A_{ab}} \hat{e}_b K_a, \quad K_a \hat{f}_b = q^{-\frac{1}{2}A_{ab}} \hat{f}_b K_a, \quad (34)$$

where A is the Cartan matrix of the affine algebra $A_{N-1}^{(1)}$.

As the first step, following [5], we rewrite $L^{(1)}(\lambda)$ and $L^{(2)}(\lambda)$ by moving $e^{\frac{\beta}{2}\Pi}$ to the extreme right,

$$L^{(1)}(\lambda) = \sum_{a=1}^N e^{-\beta\alpha_a(\Phi)} \left(\rho_+ e^{\frac{\beta}{2}\alpha_a(\Pi)} K_a \hat{e}_a + \rho_- e^{-\frac{\beta}{2}\alpha_a(\Pi)} K_a \hat{f}_a \right) e^{\beta\Pi}, \quad (35)$$

$$\begin{aligned} L^{(2)}(\lambda) = \sum_{a=1}^N & \left(\gamma_1 e^{-\beta(\alpha_a(\Phi)+\alpha_{a+1}(\Phi))} e^{\frac{\beta}{2}(\alpha_a(\Pi)+\alpha_{a+1}(\Pi))} K_a K_{a+1} \hat{e}_a \hat{e}_{a+1} \right. \\ & + \gamma_2 e^{-\beta(\alpha_a(\Phi)+\alpha_{a+1}(\Phi))} e^{-\frac{\beta}{2}(\alpha_a(\Pi)+\alpha_{a+1}(\Pi))} K_a K_{a+1} \hat{f}_{a+1} \hat{f}_a \\ & \left. + e^{-2\beta\alpha_a(\Phi)} K_a^2 (\gamma_3 \hat{e}_a \hat{f}_a + \gamma_4 \hat{f}_a \hat{e}_a) \right) e^{\beta\Pi}. \end{aligned} \quad (36)$$

Next we substitute (35)–(36) into (17)–(18) and move all the factors containing $e^{\beta\Pi}$ to the right using the relations

$$e^{\beta\Pi_1} e^{-\beta\alpha_a(\Phi)} = e^{-\beta\alpha_a(\Phi)} (K_a^2 \otimes \mathbb{I}) e^{\beta\Pi_1}, \quad e^{\beta\Pi_2} e^{-\beta\alpha_a(\Phi)} = e^{-\beta\alpha_a(\Phi)} (\mathbb{I} \otimes K_a^2) e^{\beta\Pi_2}.$$

Finally, matching the coefficients at functionally independent exponentials of quantum fields, we obtain a set of relations. Here one should take into account that $R(\lambda)$ commutes with $(e_{aa} \otimes \mathbb{I} + \mathbb{I} \otimes e_{aa})$ and hence

$$[R(\lambda), K_a \otimes K_a] = 0, \quad [R(\lambda), e^{\beta\Pi_1} e^{\beta\Pi_2}] = 0.$$

The relations that arise as matching conditions for the coefficients in (17) at the fields $e^{-\beta\alpha_a(\Phi)} e^{\pm\frac{\beta}{2}\alpha_a(\Pi)}$ are

$$R\left(\frac{\lambda}{\mu}\right) \Delta(x_a) = \Delta'(x_a) R\left(\frac{\lambda}{\mu}\right), \quad a = 1, \dots, N, \quad (37)$$

where $x_a = \hat{e}_a, \hat{f}_a$, respectively, and

$$\Delta(x_a) = x_a \otimes K_a^{-1} + K_a \otimes x_a, \quad \Delta'(x_a) = x_a \otimes K_a + K_a^{-1} \otimes x_a. \quad (38)$$

Here and below $x_N \otimes \mathbb{I}$ depends on λ while $\mathbb{I} \otimes x_N$ depends on μ .

In [5], Jimbo has shown that the solution to equations (37) is unique up to an overall scalar factor and that it is given by the R-matrix (12).

Now, treating equation (18) similarly and matching the coefficients at the fields

$$e^{-\beta(\alpha_a(\Phi)+\alpha_b(\Phi))} e^{\frac{\beta}{2}(\kappa_1\alpha_a(\Pi)+\kappa_2\alpha_b(\Pi))}, \quad \kappa_i = \pm,$$

we find the relations

$$R\left(\frac{\lambda}{\mu}\right) X_{ab}^{\kappa_1\kappa_2} = (X_{ab}^{\kappa_1\kappa_2})' R\left(\frac{\lambda}{\mu}\right), \quad a, b = 1, \dots, N, \quad (39)$$

where the prime on the r.h.s. denotes the permutation of the tensor factors (analogous to that in (38)). Obviously, $X_{ab}^{\kappa_1\kappa_2} = X_{ba}^{\kappa_2\kappa_1}$. We have

$$X_{ab}^{++} = \hat{e}_a K_b \otimes K_a^{-1} \hat{e}_b + (1 - \delta_{ab}) \hat{e}_b K_a \otimes K_b^{-1} \hat{e}_a, \quad a - b \neq \pm 1 \bmod N, \quad (40)$$

$$X_{a,a+1}^{++} = \gamma_1 (\hat{e}_a \hat{e}_{a+1} \otimes K_a^{-1} K_{a+1}^{-1} + K_a K_{a+1} \otimes \hat{e}_a \hat{e}_{a+1}) + \rho_+^2 (\hat{e}_a K_{a+1} \otimes K_a^{-1} \hat{e}_{a+1} + \hat{e}_{a+1} K_a \otimes K_{a+1}^{-1} \hat{e}_a), \quad (41)$$

$$X_{ab}^{--} = \hat{f}_a K_b \otimes K_a^{-1} \hat{f}_b + (1 - \delta_{ab}) \hat{f}_b K_a \otimes K_b^{-1} \hat{f}_a, \quad a - b \neq \pm 1 \bmod N, \quad (42)$$

$$X_{a,a+1}^{--} = \gamma_2 (\hat{f}_{a+1} \hat{f}_a \otimes K_a^{-1} K_{a+1}^{-1} + K_a K_{a+1} \otimes \hat{f}_{a+1} \hat{f}_a) + \rho_-^2 (\hat{f}_a K_{a+1} \otimes K_a^{-1} \hat{f}_{a+1} + \hat{f}_{a+1} K_a \otimes K_{a+1}^{-1} \hat{f}_a), \quad (43)$$

$$X_{ab}^{+-} = \delta_{ab} (\gamma_3 (\hat{e}_a \hat{f}_a \otimes K_a^{-2} + K_a^2 \otimes \hat{e}_a \hat{f}_a) + \gamma_4 (\hat{f}_a \hat{e}_a \otimes K_a^{-2} + K_a^2 \otimes \hat{f}_a \hat{e}_a)) + \rho_+ \rho_- (\hat{e}_a K_b \otimes K_a^{-1} \hat{f}_b + \hat{f}_b K_a \otimes K_b^{-1} \hat{e}_a). \quad (44)$$

Let us remark that eqs. (34) and (38) are relations for the generators of the affine algebra $A_{N-1}^{(1)}$ that hold for any rank and representation. However, in the case of the fundamental representation we have extra relations:

$$\begin{aligned} \text{for } N \geq 2: \quad & \hat{e}_a \hat{f}_b = \hat{f}_b \hat{e}_a = 0, \quad \text{if } b \neq a, \\ \text{for } N \geq 3: \quad & \hat{e}_a \hat{e}_b = \hat{f}_b \hat{f}_a = 0, \quad \text{if } b - a \neq 1 \bmod N. \end{aligned}$$

Taking them into account, we observe that

$$(1 + q^2) X_{aa}^{++} = \Delta(\hat{e}_a) \Delta(\hat{e}_a), \quad (1 + q^2) X_{aa}^{--} = \Delta(\hat{f}_a) \Delta(\hat{f}_a), \quad (45)$$

$$X_{ab}^{++} = \Delta(\hat{e}_a) \Delta(\hat{e}_b), \quad X_{ab}^{--} = \Delta(\hat{f}_a) \Delta(\hat{f}_b), \quad a - b \neq 0, \pm 1 \bmod N, \quad (46)$$

$$X_{ab}^{+-} = \rho_+ \rho_- \Delta(\hat{e}_a) \Delta(\hat{f}_b), \quad a \neq b, \quad (47)$$

and

$$X_{a,a+1}^{++} - \gamma_1 \Delta(\hat{e}_a) \Delta(\hat{e}_{a+1}) - q(\rho_+^2 - \gamma_1) \Delta(\hat{e}_{a+1}) \Delta(\hat{e}_a) = ((1 - q)\rho_+^2 + (q - q^{-1})\gamma_1) \hat{e}_{a+1} K_a \otimes K_{a+1}^{-1} \hat{e}_a, \quad (48)$$

$$X_{a,a+1}^{--} - \gamma_2 \Delta(\hat{f}_{a+1}) \Delta(\hat{f}_a) - q^{-1}(\rho_-^2 - \gamma_2) \Delta(\hat{f}_a) \Delta(\hat{f}_{a+1}) = ((1 - q^{-1})\rho_-^2 + (q^{-1} - q)\gamma_2) \hat{f}_a K_{a+1} \otimes K_a^{-1} \hat{f}_{a+1}, \quad (49)$$

$$X_{aa}^{+-} - \gamma_3 \Delta(\hat{e}_a) \Delta(\hat{f}_a) - \gamma_4 \Delta(\hat{f}_a) \Delta(\hat{e}_a) = (1 - \gamma_3 - \gamma_4) (\hat{e}_a K_a \otimes K_a^{-1} \hat{f}_a + \hat{f}_a K_a \otimes K_a^{-1} \hat{e}_a). \quad (50)$$

Thus the condition that $L^{(2)}$ under consideration is a solution to (18) is equivalent to the requirement that relation (39) holds for the r.h.s. of (45)–(50). Equations (37)–(38) imply that the r.h.s. of (45)–(47) does satisfy (39). Furthermore, it is straightforward to check that (39) does not hold for the r.h.s. of (48)–(50). This implies that the scalar factors on the r.h.s. of these equations must vanish. Whence we obtain the values of γ_i given in the Proposition 1.

A.2 Proof of the Proposition 1. Third order

Lemma 1. *Let $R(\lambda)$ be given by (12), and $L^{(0)}(\lambda)$ be as in (20). Let*

$$\tilde{L}^{(1)}(\lambda) = \sum_{a,b=1}^N \tilde{L}_{ab}^{(1)}(\lambda) e_{ab} \quad (51)$$

be an arbitrary solution to equation (17). Then the operator-valued coefficients $\tilde{L}_{ab}^{(1)}(\lambda)$ vanish unless $a = b$ or $a - b = \pm 1 \bmod N$.

Proof. Consider the matrix entry $e_{cb} \otimes e_{ac}$ of equation (17). Choose such a, b, c that $a \neq b$, $a \neq c$, $b \neq c$. Then, since $L^{(0)}(\lambda)$ is a diagonal matrix, the computation of the matrix element in question involves only the non-diagonal part of the R-matrix (12). It is straightforward to check that as the result we obtain the equation

$$\left(\frac{\lambda}{\mu}\right)^{\theta_{ca}} \tilde{L}_{ab}^{(1)}(\lambda) = \left(\frac{\lambda}{\mu}\right)^{\theta_{cb}} \tilde{L}_{ab}^{(1)}(\mu). \quad (52)$$

Now, if $b - a \neq 0, \pm 1 \bmod N$, then (52) for $c = a - 1 \bmod N$ and $c = a + 1 \bmod N$ yields two equations that are inconsistent unless $\tilde{L}_{ab}^{(1)}(\lambda) = 0$. This completes the proof of the Lemma. \square

In order to prove the part **ii)** of the Proposition 1, we write

$$\tilde{L}^{(2)}(\lambda) = \sum_{a,b=1}^N \tilde{L}_{ab}^{(2)}(\lambda) e_{ab}, \quad L^{(3)}(\lambda) = \sum_{a,b=1}^N L_{ab}^{(3)}(\lambda) e_{ab}. \quad (53)$$

for the general solution of (18) and the sought for solution of (19).

Consider the matrix entry $e_{a,a+1} \otimes e_{a-1,a+1}$ of equation (19) in the $N \geq 3$ case. It is straightforward to check that the resulting equation reads

$$\begin{aligned} & \left(\frac{\lambda}{\mu} - 1\right) L_{a,a+1}^{(1)}(\lambda) \tilde{L}_{a-1,a+1}^{(2)}(\mu) + (q - q^{-1}) \frac{\lambda}{\mu} \tilde{L}_{a-1,a+1}^{(2)}(\lambda) L_{a,a+1}^{(1)}(\mu) \\ &= \left(q \frac{\lambda}{\mu} - q^{-1}\right) \tilde{L}_{a-1,a+1}^{(2)}(\mu) L_{a,a+1}^{(1)}(\lambda). \end{aligned} \quad (54)$$

Note that this equation, although coming from the third order in the ε -expansion, does not involve matrix entries of $L^{(3)}$. The reason is that in (19) $L^{(3)}$ is coupled to $L^{(0)}$ for which the matrix entries $(a, a+1)$ and $(a-1, a+1)$ vanish.

Now, by Lemma 1, we can replace $\tilde{L}^{(2)}$ in (54) with the particular $L^{(2)}$ given by (22) since they must have coinciding matrix entries $(a-1, a+1)$. Finally, it is easy to check that (54) does not hold for the matrix entries of $L^{(1)}$ and $L^{(2)}$ (cf. (13) and (25)). Therefore, for any possible choice of $\tilde{L}^{(2)}$, equation (19) has no solution $L^{(3)}$. \square

A.3 Proof of Proposition 2

Part **i)**. The $L^{(1)}$ in (30) can be obtained from $L^{(1)}$ in (20) by setting $\hat{f}_N = 0$. Since relations (37) are linear in x_a , they are consistent with such a reduction. Hence it follows that $L^{(1)}$ given by (30) is a solution to (18).

Part **ii**). Analogously, setting $\hat{f}_N = 0$ in (22), we obtain (32). The direct inspection of (40)–(44) shows that X_{ab}^{++} are not affected by the reduction while X_{ab}^{--} and X_{ab}^{+-} vanish if $a = N$ or $b = N$ and do not change if $a, b \neq N$. Therefore relations (39) remain valid which, in turn, implies that (19) holds.

Part **iii**). It suffices to repeat the arguments given in Section A.2 and notice that the matrix entries $L_{a,a+1}^{(1)}$ and $L_{a-1,a+1}^{(2)}$ are not affected by the reduction. \square

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